Knot Games and Winning Strategies

Kia Braha[1] Aaron Klingensmith[1]

[1]Department of Mathematics
Seattle University

University of Washington REU, 2011

Outline

Outline

Outline

Outline

Outline

Why Games?

A large portion of knot theory research deals with:

- Finding out new information as to how knots relate to other knots
- Finding out more ways to distinguish different knots

By playing games on knots, we can better understand details or patterns that might otherwise be missed. It is our hope that these patterns lead to more concrete results, in hopes of making progress on the goals stated above.

Why Games?

A large portion of knot theory research deals with:

- Finding out new information as to how knots relate to other knots
- Finding out more ways to distinguish different knots

By playing games on knots, we can better understand details or patterns that might otherwise be missed. It is our hope that these patterns lead to more concrete results, in hopes of making progress on the goals stated above.

Why Games?

A large portion of knot theory research deals with:

- Finding out new information as to how knots relate to other knots
- Finding out more ways to distinguish different knots

By playing games on knots, we can better understand details or patterns that might otherwise be missed. It is our hope that these patterns lead to more concrete results, in hopes of making progress on the goals stated above.

Why Games?

A large portion of knot theory research deals with:

- Finding out new information as to how knots relate to other knots
- Finding out more ways to distinguish different knots

By playing games on knots, we can better understand details or patterns that might otherwise be missed. It is our hope that these patterns lead to more concrete results, in hopes of making progress on the goals stated above.

What is a Knot?

A **knot** is a closed curve in space that does not intersect itself anywhere.

A knot is unaffected by "deformities".

Thus one knot can have many different projections.

What is a Knot?

A **knot** is a closed curve in space that does not intersect itself anywhere.

A knot is unaffected by "deformities".

Thus one knot can have many different projections.

What is a Knot?

A **knot** is a closed curve in space that does not intersect itself anywhere.
A knot is unaffected by "deformities".
Thus one knot can have many different projections.

Reidemeister Moves

These ”deformities” are also known as **Reidemeister moves.**

Tricolorability

A knot is said to be **tricolorable** if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together.

Projection

A **knot projection** is simply one way, out of the many that exist, to view a knot. A knot projection does not necessarily have to have information about the crossings.

Links

A **link** is a set of knotted loops (unknots) tangled together.

The Knot Coloring Game

For this game, we use the basic rules mentioned in tricolorability, and simply assign point values.

Given a knot projection, two players take turns coloring strands in the following ways:

- Players can only color uncolored strands
- A player can only color a strand if it does not violate the rules of tricolorabiliy (i.e. it does not result in a crossing where two strands are color X and another strand is color Y)
- Each time a player completes a crossing by coloring a strand, they score a point for that crossing
- If a player colors a strand of the knot in such a way that no other strand of the knot can be colored, that player receives a bonus point for having the last move.

The Knot Coloring Game

For this game, we use the basic rules mentioned in tricolorability, and simply assign point values.
Given a knot projection, two players take turns coloring strands in the following ways:

- Players can only color uncolored strands
- A player can only color a strand if it does not violate the rules of tricolorabiliy (i.e. it does not result in a crossing where two strands are color X and another strand is color Y)
- Each time a player completes a crossing by coloring a strand, they score a point for that crossing
- If a player colors a strand of the knot in such a way that no other strand of the knot can be colored, that player receives a bonus point for having the last move.

The Knot Coloring Game

For this game, we use the basic rules mentioned in tricolorability, and simply assign point values.

Given a knot projection, two players take turns coloring strands in the following ways:

- Players can only color uncolored strands
- A player can only color a strand if it does not violate the rules of tricolorabiliy (i.e. it does not result in a crossing where two strands are color X and another strand is color Y)
- Each time a player completes a crossing by coloring a strand, they score a point for that crossing
- If a player colors a strand of the knot in such a way that no other strand of the knot can be colored, that player receives a bonus point for having the last move.

The Knot Coloring Game

For this game, we use the basic rules mentioned in tricolorability, and simply assign point values.

Given a knot projection, two players take turns coloring strands in the following ways:

- Players can only color uncolored strands
- A player can only color a strand if it does not violate the rules of tricolorabiliy (i.e. it does not result in a crossing where two strands are color X and another strand is color Y)
- Each time a player completes a crossing by coloring a strand, they score a point for that crossing
- If a player colors a strand of the knot in such a way that no other strand of the knot can be colored, that player receives a bonus point for having the last move.

The Knot Coloring Game

For this game, we use the basic rules mentioned in tricolorability, and simply assign point values.

Given a knot projection, two players take turns coloring strands in the following ways:

- Players can only color uncolored strands
- A player can only color a strand if it does not violate the rules of tricolorabiliy (i.e. it does not result in a crossing where two strands are color X and another strand is color Y)
- Each time a player completes a crossing by coloring a strand, they score a point for that crossing
- If a player colors a strand of the knot in such a way that no other strand of the knot can be colored, that player receives a bonus point for having the last move.

Here is an example of an allowable game and a game that includes a violation.

Knotting vs. Unknotting Game

For this game, two players take turns deciding on the crossing orientations of a knot where all the crossings are originally undetermined.

- At the start of the game, one player will choose to be the "knotter" and the other player will take the role of the "unknotter".
- The goal of the knotter is to change the crosses in such a way that the final knots results in an actual knot. The unknotter's goal is to keep a knot from being formed.
- The winner is determined after a knot has obviously been created, or it is clear that no knot can be created.

Knotting vs. Unknotting Game

For this game, two players take turns deciding on the crossing orientations of a knot where all the crossings are originally undetermined.

- At the start of the game, one player will choose to be the "knotter" and the other player will take the role of the "unknotter".
- The goal of the knotter is to change the crosses in such a way that the final knots results in an actual knot. The unknotter's goal is to keep a knot from being formed.
- The winner is determined after a knot has obviously been created, or it is clear that no knot can be created.

Knotting vs. Unknotting Game

For this game, two players take turns deciding on the crossing orientations of a knot where all the crossings are originally undetermined.

- At the start of the game, one player will choose to be the "knotter" and the other player will take the role of the "unknotter".
- The goal of the knotter is to change the crosses in such a way that the final knots results in an actual knot. The unknotter's goal is to keep a knot from being formed.
- The winner is determined after a knot has obviously been created, or it is clear that no knot can be created.

Knotting vs. Unknotting Game

For this game, two players take turns deciding on the crossing orientations of a knot where all the crossings are originally undetermined.

- At the start of the game, one player will choose to be the "knotter" and the other player will take the role of the "unknotter".
- The goal of the knotter is to change the crosses in such a way that the final knots results in an actual knot. The unknotter's goal is to keep a knot from being formed.
- The winner is determined after a knot has obviously been created, or it is clear that no knot can be created.

Determining a Winning Strategy

The main goal is to determine if there is, in fact, a winning strategy for either player.

- Initially, we chose a relatively simple family of knots to play the tricolorability game to see if we could gain some intuition as to what a strategy might look like.
- The family of knots that seemed to lend itself to some initial trial and error was **torus knots** as they have some nice symmetry and are relatively simple.
- After playing a considerable number of games on torus knots, we were beginning to notice that the first player to color a knot could usually win.

Determining a Winning Strategy

The main goal is to determine if there is, in fact, a winning strategy for either player.

- Initially, we chose a relatively simple family of knots to play the tricolorability game to see if we could gain some intuition as to what a strategy might look like.
- The family of knots that seemed to lend itself to some initial trial and error was **torus knots** as they have some nice symmetry and are relatively simple.
- After playing a considerable number of games on torus knots, we were beginning to notice that the first player to color a knot could usually win.

Determining a Winning Strategy

The main goal is to determine if there is, in fact, a winning strategy for either player.

- Initially, we chose a relatively simple family of knots to play the tricolorability game to see if we could gain some intuition as to what a strategy might look like.
- The family of knots that seemed to lend itself to some initial trial and error was **torus knots** as they have some nice symmetry and are relatively simple.
- After playing a considerable number of games on torus knots, we were beginning to notice that the first player to color a knot could usually win.

Determining a Winning Strategy

The main goal is to determine if there is, in fact, a winning strategy for either player.

- Initially, we chose a relatively simple family of knots to play the tricolorability game to see if we could gain some intuition as to what a strategy might look like.
- The family of knots that seemed to lend itself to some initial trial and error was **torus knots** as they have some nice symmetry and are relatively simple.
- After playing a considerable number of games on torus knots, we were beginning to notice that the first player to color a knot could usually win.

However, torus knots quickly became too big to work out all the cases by hand, and we wanted to determine if we could find a way to embed smaller games into the larger ones to support our hypothesis.

This idea posed a large challenge, and in order to continue in the direction we were heading, we needed some simpler ways to represent the knots.

However, torus knots quickly became too big to work out all the cases by hand, and we wanted to determine if we could find a way to embed smaller games into the larger ones to support our hypothesis.
This idea posed a large challenge, and in order to continue in the direction we were heading, we needed some simpler ways to represent the knots.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.

This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.
This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.
This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.
This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.
This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Because knot projections are not the easiest items to work with when constructing a proof, much of our early research was devoted to finding new ways to represent knots.
This lead to the following ways of representing knots:

- Sets of numbers
- (Sometimes) Directed graphs
- "Lines" (but more on this later)

Nothing was presenting itself nicely, however, and it was at this time that our third research partner presented us with a push forward from another direction.

Up to this point, all we had was a hypothesis that the player with a winning strategy was influenced by how many crossings there were for a knot.

What our partner, Everett Sullivan, was able to do was create a program that played our knot game optimally for both players on a given inputted knot, and display who would win.

This confirmed our hypothesis on every knot up to nine crossings, encouraging us, but we still lacked a proof, and there were some issues presented by the programs.

Up to this point, all we had was a hypothesis that the player with a winning strategy was influenced by how many crossings there were for a knot.

What our partner, Everett Sullivan, was able to do was create a program that played our knot game optimally for both players on a given inputted knot, and display who would win.

This confirmed our hypothesis on every knot up to nine crossings, encouraging us, but we still lacked a proof, and there were some issues presented by the programs.

Up to this point, all we had was a hypothesis that the player with a winning strategy was influenced by how many crossings there were for a knot.

What our partner, Everett Sullivan, was able to do was create a program that played our knot game optimally for both players on a given inputted knot, and display who would win.

This confirmed our hypothesis on every knot up to nine crossings, encouraging us, but we still lacked a proof, and there were some issues presented by the programs.

So in our search to make sense of this pattern, we began to more thoroughly examine the roles of Reidemeister moves.
In essence, one can think of a knot as an unknot that has been deformed by Reidemeister moves and crossing changes.
What we found was that neither crossing changes, nor the second and third types of Reidemesiter moves affect whether a knot has an even or odd number of crossings.

So in our search to make sense of this pattern, we began to more thoroughly examine the roles of Reidemeister moves.

In essence, one can think of a knot as an unknot that has been deformed by Reidemeister moves and crossing changes.

What we found was that neither crossing changes, nor the second and third types of Reidemesiter moves affect whether a knot has an even or odd number of crossings.

So in our search to make sense of this pattern, we began to more thoroughly examine the roles of Reidemeister moves. In essence, one can think of a knot as an unknot that has been deformed by Reidemeister moves and crossing changes. What we found was that neither crossing changes, nor the second and third types of Reidemesiter moves affect whether a knot has an even or odd number of crossings.

Reidemeister One does.

This led us to questions: When looking at knots with a crossing number greater than two, could the minimum number of type one moves done when turning an unknot into a knot be some sort of invariant? Possibly more information?

Reidemeister One does.

This led us to questions: When looking at knots with a crossing number greater than two, could the minimum number of type one moves done when turning an unknot into a knot be some sort of invariant? Possibly more information?

This naturally led us to want to examine twist knots.
By going down this path, we noticed our pattern was still holding, and were able to come up with a result and proof.

Given an unknot created from a simple loop using $2n-1$ Reidemeister I moves, there exists a winning strategy for player one.

This naturally led us to want to examine twist knots.
By going down this path, we noticed our pattern was still holding, and were able to come up with a result and proof.

Given an unknot created from a simple loop using $2n-1$ Reidemeister I moves, there exists a winning strategy for player one.

This naturally led us to want to examine twist knots.
By going down this path, we noticed our pattern was still holding, and were able to come up with a result and proof.

Given an unknot created from a simple loop using $2n - 1$ Reidemeister I moves, there exists a winning strategy for player one.

We were also able to come up with other proofs, that we'd be more than willing to share with anyone who is curious, but are currently in the process of formalizing them:

- Given $2n$ disjoint trefoil knots, player two has a winning strategy.
- Given $2n + 1$ disjoint trefoil knots, player one has a winning strategy.
- Given an unknot with K Reidemeister I moves, where $K > 1, K \in \mathbb{Z}$ and the coloring game is played with only two colors used, then at least one strand will remain uncolored.